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Characteristic cycle of a rank 1 sheaf on a surface: research announcement

By

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Abstract

This is a research announcement about a study of characteristic cycle of a rank 1 sheaf on a surface on which I am writing a paper. In this announcement, we construct a canonical lifting of Kato's logarithmic characteristic cycle on the cotangent bundle of the surface. As a corollary, an index formula computing the Euler characteristic of the sheaf is yielded by the canonical lifting.

§ 1. Introduction

This is a research announcement about a study of characteristic cycle of a rank 1 sheaf on a surface on which I am writing a paper. We state without proof a result on the characteristic cycle of a smooth sheaf of rank 1 on a surface of positive characteristic.

Let X be a smooth separated connected scheme of dimension d over an algebraically closed field k of characteristic $p > 0$. Let $\mathcal{F}$ be a constructible complex of Λ-modules on X, where Λ is a finite filed of characteristic $\ell \neq p$. A constructible complex $\mathcal{F}$ is a complex of étale sheaves such that $\mathcal{H}^q(\mathcal{F})$ is constructible for any q, and equal to 0 except for finitely many q.

The characteristic cycle of $\mathcal{F}$ is an analogue of that of a holonomic $\mathcal{D}$-module on a smooth variety of characteristic 0 in the theory of $\mathcal{D}$-modules. It is defined as a d-cycle on the cotangent bundle T^*X of X. The cotangent bundle T^*X of X is the vector bundle on X corresponding to Ω^1_X. The characteristic cycle of $\mathcal{F}$ satisfies an index

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formula computing the Euler characteristic

$$\chi(X,\mathcal{F})=\sum_{i=0}^{2d}(-1)^i\dim H^i(X,\mathcal{F})$$

of $\mathcal{F}$.

We see a classical example of characteristic cycle. We assume that $d=1$. Let U be the complement of a divisor D on X and $j\colon U\to X$ the canonical open immersion. We assume that $\mathcal{F}$ is the zero extension $j_!\mathcal{G}$ of a smooth sheaf $\mathcal{G}$ of Λ-modules over U. Let T^*_XX (resp. T^*_xX) denote the zero-section of T^*X (resp. the fiber of T^*X at a closed point x of X). For an integral closed subscheme C of T^*X, we write $[C]$ for C as a prime cycle on T^*X. Then the characteristic cycle $\mathrm{Char}(\mathcal{F})$ of $\mathcal{F}$ is defined by

$$\mathrm{Char}\,(\mathcal{F})=-\left(\mathrm{rank}\,(\mathcal{G})\,[T^*_XX]+\sum_{x\in D}(\mathrm{rank}\,(\mathcal{G})+\mathrm{Sw}_x\mathcal{G})\,[T^*_xX]\right). \tag{1.1}$$

In (1.1), the symbol $\mathrm{Sw}_x\mathcal{G}$ is an invariant of ramification called the Swan conductor of $\mathcal{G}$ at x. The Swan conductor of $\mathcal{G}$ is a non-negative integer and measures the wild ramification of $\mathcal{G}$. The index formula in this case is the classical Grothendieck-Ogg-Shafarevich formula ([SGA5]). That is, if X is proper, then

$$\chi(X,\mathcal{F})=(\mathrm{Char}(\mathcal{F}),T^*_XX)_{T^*X},$$

where the right hand side denotes the intersection number in T^*X.

In the general dimensional case, the characteristic cycle of a constructible complex $\mathcal{F}$ is defined by T. Saito using Beilinson's singular support ([B]) and vanishing cycles in [S4]. The index formula yielded by this characteristic cycle generalizes Deligne and Laumon's formula for the Euler characteristic for surfaces ([L] Théorème 1.2.1). However, this characteristic cycle is hard to compute in general.

In the case where $d=2$, let U be the complement of a divisor D on X with simple normal crossings, and $j\colon U\to X$ the canonical open immersion. We assume that $\mathcal{F}$ is the zero extension $j_!\mathcal{G}$ of a smooth sheaf $\mathcal{G}$ of Λ-modules of rank 1 over U. With this setting, Kato has given another definition of characteristic cycle on the logarithmic cotangent bundle of X with logarithmic poles D using ramification theory ([K2]). This characteristic cycle seems easier to compute. We denote it by $\mathrm{Char}(X,U,\mathcal{G})$. The index formula computing the Euler characteristic $\chi(X,\mathcal{F})$ as the intersection number of this cycle with the zero-section $T^*_XX(\log D)\subset T^*X(\log D)$ is proved by Kato ([S1]).

We keep the assumption in the last paragraph. The main result in this announcement is a construction of a 2-cycle on the cotangent bundle T^*X of X which is a canonical lifting of Kato's characteristic cycle (Theorem 3.2). For the construction of this canonical lifting, we use Matsuda's non-logarithmic ramification theory ([M]). Matsuda's theory is non-logarithmic version of the ramification theory which Kato used. We expect that the canonical lifting is equal to Saito's characteristic cycle (Conjecture 4.3).

In this announcement, we assume that $p \neq 2$ for simplicity. In the $p = 2$ case, a new interesting phenomenon arises, which we will discuss in the paper which I am writing.

Throughout this announcement, let k be an algebraically closed field of characteristic $p \geq 3$ and X a smooth separated connected scheme of dimension 2 over k. We write D for a divisor on X with simple normal crossings, and put $U = X - D$. The symbol Λ denotes a finite field of characteristic $\ell \neq p$. We consider a smooth sheaf $\mathcal{G}$ of Λ-modules of rank 1 on U corresponding to a character $\chi\colon \pi_1^{\mathrm{ab}}(U) \to \Lambda^\times$. We fix an inclusion $\Lambda^\times \hookrightarrow \mathbb{Q}/\mathbb{Z}$ and identify χ with an element of $H^1(U, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(\pi_1^{\mathrm{ab}}(U), \mathbb{Q}/\mathbb{Z})$. We put $\mathcal{F} = j_!\mathcal{G}$, where $j\colon U \to X$ is the canonical open immersion.

§ 2. Kato's logarithmic characteristic cycle

In this section, we recall Kato's logarithmic characteristic cycle. Kato's logarithmic characteristic cycle is defined as a 2-cycle on the logarithmic cotangent bundle $T^*X(\log D)$ of X with logarithmic poles along D. This cycle satisfies an index formula computing the Euler characteristic.

Let $\{D_i\}_{i\in I}$ be the irreducible components of D and let $\mathfrak{p}_i$ denote the generic point of D_i. The local field at $\mathfrak{p}_i$ means the complete discrete valuation field $\mathrm{Frac}\,\hat{\mathcal{O}}_{X,\mathfrak{p}_i}$, where $\hat{\mathcal{O}}_{X,\mathfrak{p}_i}$ denotes the completion of the local ring $\mathcal{O}_{X,\mathfrak{p}_i}$ at $\mathfrak{p}_i$ by the maximal ideal. We denote the local field at $\mathfrak{p}_i$ by K_i. Let $\chi|_{K_i}$ denote the image of χ by the composition of canonical maps

$$H^1(U, \mathbb{Q}/\mathbb{Z}) \to H^1(k(X), \mathbb{Q}/\mathbb{Z}) \to H^1(K_i, \mathbb{Q}/\mathbb{Z}),$$

where $k(X)$ denotes the function field of X.

We consider the ramification filtration $\{\mathrm{fil}_n H^1(K_i, \mathbb{Q}/\mathbb{Z})\}_{n\geq 0}$ of $H^1(K_i, \mathbb{Q}/\mathbb{Z})$ defined in [K1] Definition (2.1). We define the Swan conductor $\mathrm{sw}(\chi|_{K_i})$ to be the minimal number n such that $\chi|_{K_i} \in \mathrm{fil}_n H^1(K_i, \mathbb{Q}/\mathbb{Z})$. We put $R_\chi = \sum_{i\in I} \mathrm{sw}(\chi|_{K_i}) D_i$, and call it the Swan conductor divisor of χ on X. This is an effective Cartier divisor on X. Let Z denote the support of R_χ. For $D_i \subset Z$, we define the refined Swan conductor $\mathrm{rsw}(\chi|_{K_i})$ to be the image of $\chi|_{K_i}$ by the map $\mathrm{gr}_{\mathrm{sw}(\chi|_{K_i})} H^1(K_i, \mathbb{Q}/\mathbb{Z}) \to (\Omega^1_X(\log D)(R_\chi)|_Z)_{\mathfrak{p}_i}$ defined in [M] Remark 3.2.12.

Lemma 2.1 ([K2] (3.4.2)). *There exists a unique global section* $\mathrm{rsw}(\chi)$ *of the sheaf* $\Omega^1_X(\log D)(R_\chi)|_Z$ *whose germ* $\mathrm{rsw}(\chi)_{\mathfrak{p}_i}$ *at any generic point* $\mathfrak{p}_i$ *of* Z *coincides with the refined Swan conductor* $\mathrm{rsw}(\chi|_{K_i})$.

Let $T^*X(\log D) = \operatorname{Spec} \mathbb{V}(\Omega^1_X(\log D)^\vee)$ denote the logarithmic cotangent bundle of X with logarithmic poles along D. Kato introduced the notion of cleanness ([K2] (3.4.3)). We define a non-negative integer $\operatorname{ord}_\chi(x, D_i)$ for a point x of Z and an irreducible component D_i of Z containing x by

$$\operatorname{ord}_\chi(x, D_i) = \max\{n \in \mathbb{Z}_{\geq 0}\ ;\ \operatorname{rsw}(\chi)|_{D_i,x} \in m_x^n \Omega^1_X(\log D)(R_\chi)|_{D_i,x}\}.$$

Here m_x is the maximal ideal of the local ring $\mathcal{O}_{X,x}$ at x. We say that $(X, U, \mathcal{G})$ is clean at a point x of X if $x \notin Z$ or if $x \in Z$ and $\operatorname{ord}_\chi(x, D_i) = 0$ for an irreducible component D_i of Z containing x. We say that $(X, U, \mathcal{G})$ is clean if $(X, U, \mathcal{G})$ is clean at all points of X. He defined a logarithmic characteristic cycle $\operatorname{Char}(X, U, \mathcal{G})$ of $(X, U, \mathcal{G})$ as a 2-cycle of $T^*X(\log D)$ using the refined Swan conductor $\operatorname{rsw}(\chi)$ above as follows ([K2] (3.4.4)).

Let $T^*_X X(\log D)$ be the zero-section of $T^*X(\log D)$, and let $T^*_x X(\log D)$ be the fiber at a closed point x of X. We define a 2-dimensional integral closed subscheme L_i of $T^*X(\log D)$ for D_i contained in Z to be the sub line bundle of $T^*X(\log D) \times_X D_i$ associated to the unique locally direct factor of rank 1 of $\Omega^1_X(\log D)|_{D_i}$ containing the image of the multiplication map

$$\mathcal{O}_X(-R_\chi)|_{D_i} \to \Omega^1_X(\log D)|_{D_i}; \quad f \mapsto f\operatorname{rsw}(\chi).$$

For D_i not contained in Z, we define L_i by $L_i = \emptyset$.

Then the logarithmic characteristic cycle $\operatorname{Char}(X, U, \mathcal{G})$ is of the form

$$\operatorname{Char}(X, U, \mathcal{G}) = [T^*_X X(\log D)] + \sum_{i \in I} \operatorname{sw}(\chi|_{K_i})[L_i] + \sum_{x \in |D|} s_x [T^*_x X(\log D)], \tag{2.1}$$

where $|D|$ denotes the set of closed points of D. For the definition of s_x in (2.1), we take a composition $f\colon X' = X_s \to X_{s-1} \to \cdots \to X_0 = \operatorname{Spec} \mathcal{O}_{X,x}$ of blowing-ups at closed points lying over x such that $(X', f^{-1}(U), f^*\mathcal{G})$ is clean ([K2] Theorem 4.1). We put $D' = (f^{-1}(D))_{\mathrm{red}}$. Then D' is a divisor on X' with simple normal crossings. We define $r_x \in \mathbb{Z}$ by $r_x = -(R_{\chi'} - f^*R_\chi, R_{\chi'} + K_{X'} + D + f^*R_\chi)$, where χ' denotes the pull-back of χ to $H^1(f^{-1}(U), \mathbb{Q}/\mathbb{Z})$ and $R_{\chi'}$ denotes the Swan conductor divisor of χ' on X' ([K2] Remark 5.7). We define s_x by

$$s_x = \sum_{\substack{i \in I \\ x \in D_i}} \operatorname{sw}(\chi|_{K_i}) \operatorname{ord}_\chi(x, D_i) - r_x.$$

If $(X, U, \mathcal{G})$ is clean at x, the integer s_x is equal to 0 by the definition of s_x.

The following theorem follows from [S1] the remark right after the conjecture in the page 168 and the definition of $\operatorname{Char}(X, U, \mathcal{G})$.

Theorem 2.2 (Index formula). *If X is proper over k, we have*

$$\chi(X, \mathcal{F}) = (\operatorname{Char}(X, U, \mathcal{G}), T^*_X X(\log D))_{T^*X(\log D)}.$$

§ 3. Construction of a canonical lifting

In this section, we construct a 2-cycle on T^*X which is a canonical lifting of Kato's characteristic cycle using Matsuda's non-logarithmic ramification theory ([M]). This is the main result in this announcement (Theorem 3.2). As a corollary, we have an index formula yielded by the canonical lifting. For simplicity, we assume that $(X, U, \mathcal{G})$ is clean. For the general case, we will discuss in the paper which I am writing.

We consider another filtration $\{\mathrm{fil}'_n H^1(K_i, \mathbb{Q}/\mathbb{Z})\}_{n\geq 0}$ of $H^1(K_i, \mathbb{Q}/\mathbb{Z})$ ([M] 3.1). We define a conductor $\mathrm{sw}'(\chi|_{K_i})$ as the minimal number n such that $\chi|_{K_i} \in \mathrm{fil}'_n H^1(K_i, \mathbb{Q}/\mathbb{Z})$. We put $R'_\chi = Z + \sum_{i\in I} \mathrm{sw}'(\chi|_{K_i}) D_i$. This is an effective Cartier divisor on X. For $D_i \subset Z$, we define the non-logarithmic version $\mathrm{rsw}'(\chi|_{K_i})$ of the refined Swan conductor of $\chi|_{K_i}$ to be the image of $\chi|_{K_i}$ by the map

$$\mathrm{gr}'_{\mathrm{sw}'(\chi|_{K_i})} H^1(K_i, \mathbb{Q}/\mathbb{Z}) \to \left(\Omega^1_X \left(R'_\chi\right)|_Z\right)_{\mathfrak{p}_i}$$

defined in [M] Definition 3.2.5.

Lemma 3.1 ([M] 5.2)**.** *There exists a unique global section* $\mathrm{rsw}'(\chi)$ *of the sheaf* $\Omega^1_X(R_\chi)|_Z$ *whose germ* $\mathrm{rsw}'(\chi)_{\mathfrak{p}_i}$ *at any generic point* $\mathfrak{p}_i$ *of* Z *coincides with the refined Swan conductor* $\mathrm{rsw}'(\chi|_{K_i})$*.*

Let $T^*X = \mathrm{Spec}\, \mathbb{V}(\Omega^{1\vee}_X)$ denote the cotangent bundle of X. We define a 2-cycle $\mathrm{Char}'(X, U, \mathcal{G})$ on T^*X, which will be a canonical lifting of Kato's characteristic cycle, as follows. Let $T^*_X X$ denote the zero section of T^*X. Let $T^*_{D_i} X$ denote the conormal bundle of D_i in X, and $T^*_x X$ the fiber at a closed point x of X. We define a 2-dimensional integral closed subscheme L'_i of T^*X for D_i contained in Z to be the sub line bundle of $T^*X \times_X D_i$ associated to the unique locally direct factor of rank 1 of $\Omega^1_X|_{D_i}$ containing the image of the multiplication map

$$\mathcal{O}_X(-R'_\chi)|_{D_i} \to \Omega^1_X|_{D_i}; \quad f \mapsto f\mathrm{rsw}'(\chi).$$

For D_i not contained in Z, we define $L'_i = T^*_{D_i} X$. We put $R''_\chi = D + \sum_{i\in I} \mathrm{sw}'(\chi|_{K_i}) D_i$. Let $\mathrm{dt}(\chi|_{K_i})$ denote the multiplicity of D_i in R''_χ.

We construct a 2-cycle $\mathrm{Char}'(X, U, \mathcal{G})$ of the form

$$\mathrm{Char}'(X, U, \mathcal{G}) = [T^*_X X] + \sum_{i\in I} \mathrm{dt}(\chi|_{K_i})[L'_i] + \sum_{x\in|D|} t_x [T^*_x X]. \tag{3.1}$$

We define the integer t_x in (3.1) as follows. For a point x of Z and an irreducible component D_i of Z containing x, we define a non-negative integer $\mathrm{ord}'_\chi(x, D_i)$ by

$$\mathrm{ord}'_\chi(x, D_i) = \max\{n \in \mathbb{Z}_{\geq 0}\ ;\ \mathrm{rsw}(\chi)|_{D_i,x} \in m^n_x \Omega^1_X(R'_\chi)|_{D_i,x}\},$$

where m_x is the maximal ideal of the local ring $\mathcal{O}_{X,x}$ at x. Let x be a closed point of D. We define t_x by

$$t_x = \sharp(T_x) - 1 + \sum_{D_i \in T'_x} \mathrm{sw}(\chi|_{K_i})(\mathrm{ord}'_\chi(x, D_i) + \sharp(T_x) - \sharp(T'_x)) + \delta_{\mathrm{sw}(\chi|_{K_i})\mathrm{dt}(\chi|_{K_i})}(1 - \sharp(T_x)),$$

where $T_x = \{D_i \subset D \mid x \in D_i\}$ and $T'_x = \{D_i \in T_x \mid \mathrm{sw}(\chi|_{K_i}) > 0\}$. The symbol $\delta_{\mathrm{sw}(\chi|_{K_i})\mathrm{dt}(\chi|_{K_i})}$ is the Kronecker delta.

Let $\pi\colon T^*X \to T^*X(\log D)$ be the canonical morphism of vector bundles on X. Let $SS(X, U, \mathcal{G}) \subset T^*X(\log D)$ denote the support of (2.1). Let $SS'(X, U, \mathcal{G}) \subset T^*X$ denote the support of (3.1). Then it follows that $SS'(X, U, \mathcal{G}) \subset \pi^{-1}(SS(X, U, \mathcal{G}))$. Let $\pi^!\colon CH_2(SS(X, U, \mathcal{G})) \to CH_2(\pi^{-1}(SS(X, U, \mathcal{G})))$ be the refined Gysin homomorphism for the l.c.i. morphism π ([F] 6.6). The following theorem is the main result in this announcement.

Theorem 3.2. *The image of (3.1) in $CH_2(\pi^{-1}(SS(X, U, \mathcal{G})))$ is equal to the image of* (2.1) *by $\pi^!\colon CH_2(SS(X, U, \mathcal{G})) \to CH_2(\pi^{-1}(SS(X, U, \mathcal{G})))$.*

The next corollary follows from Theorem 3.2 since Kato's characteristic cycle (2.1) satisfies the index formula.

Corollary 3.3 (Index formula). *If X is proper over k, we have*

$$\chi(X, \mathcal{F}) = (\mathrm{Char}'(X, U, \mathcal{G}), T^*_X X)_{T^*X}.$$

§ 4. Saito's non-logarithmic characteristic cycle and a conjecture

Saito has given a definition of non-logarithmic characteristic cycle of a constructible complex on a smooth variety of general dimension using Beilinson's singular support ([B]) in [S4]. This characteristic cycle is characterized by the Milnor formula and satisfies an index formula ([S4]). The index formula for this characteristic cycle is a generalization of Deligne and Laumon's formula for the Euler characteristic for surfaces ([L] Théorème 1.2.1).

We keep the assumption in §2 and §3. Saito's non-logarithmic characteristic cycle under this assumption is equal to that defined in [S3] ([S4] Theorem 7.14). In this section, we briefly recall Saito's non-logarithmic characteristic cycle defined in ([S3]) without giving the detail of the construction. At the end of this section, we state a conjecture on the equality of Saito's characteristic cycle and the canonical lifting of Kato's characteristic cycle constructed in the previous section.

We keep the notation in §3. The divisor R''_χ on X is shown to be equal to the slope R of $\mathcal{G}$ ([S2] Definition 3.1) similarly as in the proof of Théorème 9.10 in [AS].

Saito's non-logarithmic characteristic cycle $\mathrm{Char}^{\mathcal{R}}(\mathcal{F})$ in this case is a 2-cycle on T^*X of the form

$$\mathrm{Char}^{\mathcal{R}}(\mathcal{F}) = [T^*_X X] + \sum_{i\in I} \mathrm{dt}(\chi|_{K_i})[L'_i] + \sum_{x\in|D|} u_x[T^*_x X] \tag{4.1}$$

([S2] Definition 3.5, [S3] Definition 3.8, 3.15, and Proposition 3.19). The Milnor formula characterizing this characteristic cycle is Theorem 4.1 below. Let $f\colon X \to C$ be a flat morphism to a smooth curve C over k. Let df denote the section of T^*X defined by the image of a basis of $T^*C \times_C X$ by the canonical morphism $T^*C \times_C X \to T^*X$ induced by f. The condition that f is non-characteristic with respect to $\mathcal{F}$ is introduced by Saito ([S3] Section 1). This means that the intersection of df and the support of the characteristic cycle is empty.

Theorem 4.1 (Milnor formula, [S3] Theorem 3.17)**.** *Let $f\colon X \to C$ be a flat morphism to a smooth curve over k. Let x be a closed point of X. Assume that f is non-characteristic with respect to $\mathcal{F}$ in a neighborhood of x possibly except for x and that D is étale over C. Then we have*

$$-\dim\ \mathrm{tot}\phi_x(\mathcal{F}, f) = (\mathrm{Char}^{\mathcal{R}}(\mathcal{F}), [df])_{T^*X,x}, \tag{4.2}$$

where $\dim\ \mathrm{tot}\phi_x(\mathcal{F}, f)$ *denotes the total dimension of the space $\phi_x(\mathcal{F}, f)$ of the vanishing cycles at x, and the right hand side means the intersection number in the fiber of T^*X at x.*

Saito's characteristic cycle satisfies the index formula.

Theorem 4.2 (Index formula, [S3] Theorem 3.19.)**.** *If X is proper over k, we have*

$$\chi(X, \mathcal{F}) = (\mathrm{Char}^{\mathcal{R}}(\mathcal{F}), T^*_X X)_{T^*X}.$$

Finally, we state a conjecture.

Conjecture 4.3. *Assume that $(X, U, \mathcal{G})$ is clean. Then the characteristic cycles* $\mathrm{Char}^{\mathcal{R}}(\mathcal{F})$ *and* $\mathrm{Char}'(X, U, \mathcal{G})$ *are equal.*

In order to prove this conjecture, by (3.1) and (4.1), it is sufficient to prove the equality of t_x in (3.1) and u_x in (4.1) for any closed point x on D. The equality of the sum of t_x and that of u_x follows from the index formulas Corollary 3.3 and Theorem 4.2. When the sheaf $\mathcal{G}$ is Artin-Schreier, this conjecture is proved using this equality and the fact ([S3] Corollary 3.15) that u_x is determined étale locally.

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